Restricted Mesh Simplification Using Edge Contractions

Mattias Andersson
Department of Computer Science, Lund University
Box 118, 221 00 Lund, Sweden
mattias@cs.lth.se

Joachim Gudmundsson
*National ICT Australia Ltd**
Locked Bay 9013, Alexandria NSW 1435, Australia
Joachim.Gudmundsson@nicta.com.au

Christos Levcopoulos
Department of Computer Science, Lund University
Box 118, 221 00 Lund, Sweden
christos@cs.lth.se

Received November 15, 2006
Revised October 9, 2007
Communicated by Danny Chen and D. T. Lee

We consider the problem of simplifying a planar triangle mesh using edge contractions, under the restriction that the resulting vertices must be a subset of the input set. That is, contraction of an edge must be made onto one of its adjacent vertices, which results in removing the other adjacent vertex. We show that if the perimeter of the mesh consists of at most five vertices, then we can always find a vertex not on the perimeter which can be removed in this way. If the perimeter consists of more than five vertices such a vertex may not exist. In order to maintain a higher number of removable vertices under the above restriction, we study edge flips which can be performed in a visually smooth way. A removal of a vertex which is preceded by one such smooth operation is called a 2-step removal. Moreover, we introduce the possibility that the user defines "important" vertices (or edges) which have to remain intact. Given m such important vertices, or edges, we show that a simplification hierarchy of size $O(n)$ and depth $O(\log(n/m))$ can be constructed by 2-step removals in $O(n)$ time, such that the simplified graph contains the m important vertices and edges, and at most $O(m)$ other vertices from the input graph. In some triangulations, many vertices may not even be 2-step removable. In order to provide the option to remove such vertices, we also define and examine k-step removals. This increases the lower bound on the number of removable vertices.

Keywords: Computational Geometry, Computer graphics, Edge Contractions.

---

*NICTA is funded by the Australian Government's Backing Australia's Ability initiative, in part through the Australian Research Council.

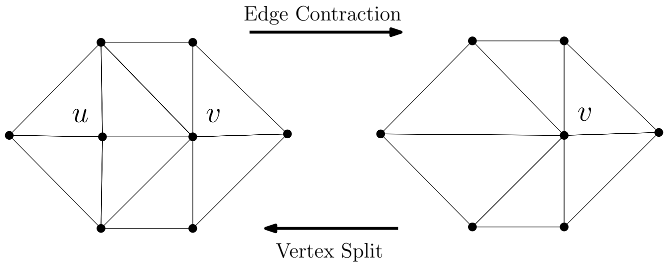


Fig. 1. An edge contraction, and its inverse operation a vertex split.

1. Introduction

In computer graphics, objects are commonly represented using triangle meshes. One important problem regarding these meshes is how to efficiently simplify them, while maintaining a good approximation of the original mesh. As an example, scanners often produce information-redundant meshes containing millions of vertices and triangles. Further, often the simplification should be performed in several rounds, such that a level-of-detail hierarchy is constructed. One application of such a hierarchy is that an appropriate level may be chosen depending on viewing distance, as finer details tend to be unnecessary as the distance increases. Other applications include progressive transmission and efficient storing.

It is common to represent the level-of-detail hierarchy as a directed, acyclic and hierarchical graph, where each level in the graph corresponds to a level in the level-of-detail hierarchy, and where each node in the graph corresponds to a triangle. The first, top-most, level in the graph corresponds to the input mesh. When a contraction is made two triangles disappear, and one or more triangles are affected in such a way that their appearance change. In the graph this is represented with edges between disappearing triangles at some level i, and the affected triangles at level $i + 1$. The efficiency of a simplification algorithm is directly related to the size [7,15] and depth of the hierarchy graph that it produces. Simplification algorithms constructing hierarchies of size $O(n)$ and depth $O(\log n)$ have been presented for several problem variants [4,6,8,14].

Another related problem [1,3] is where a triangulation is not simplified but transformed into a different triangulation, possibly on a set of different points. The transformation is done using edge flips and point moves.

Mesh simplification is generally regarded as a mature field (see [9] for a survey), consisting of several suggested methods and problem variants. In this paper we consider the method of iteratively contracting edges [4,11,10,13], where contractions are made such that no edge crossings occur during the process. Many of the results in previous papers were achieved using such standard edge contractions.

In this paper we impose a new restriction, namely that an edge must be contracted onto one of its end vertices, see Fig. 1. In order to achieve results under this restriction we concentrate on the planar setting, which is suitable for modeling terrains in 3D. The effect of such a contraction is that one of its end vertices is removed. We call vertices that can be removed in this simple way 1-step removable.

It would be preferable to reduce the size of the triangulation by identifying only 1-step removable vertices. For this purpose, we show that in the planar setting there is always at least one 1-step removable vertex inside every non-empty cycle of length smaller than six, see Theorem 2. This can be applicable, for example, within rectangular frames which may represent rectangular windows. It follows that in such cases one can proceed by repeatedly reducing the number of vertices in the interior by simple 1-step removals. However, this result does not suffice to ensure a hierarchical graph of logarithmic size, nor does it guarantee that the user can specify a substantial amount of vertices (and/or edges) he wants to keep intact, while still being able to perform simple edge contractions. To be able to guarantee both these desired options, we introduce a smooth alternative to 1-step reductions, namely 2-step removals. They result in a small modification of the mesh around the vertex to be removed. A 2-step removal is also geomorphic, i.e., it is visually smooth, it avoids degenerate intermediate triangles of zero area, and it involves at most two straight line movements. We call vertices which can be removed in this way 2-step removable. We also introduce the possibility that the user defines “important” vertices (or edges) which have to remain intact. Given m such important points we show that a hierarchical graph of size $O(n)$ and depth $O(\log(n/m))$ can be achieved using 2-step removals in linear time, while maintaining the m important vertices.

Some vertices are not even 2-step removable, and thus, we also define and study k-step removable vertices. These are also visually smooth operations, like 2-step removals, with the only difference that they may involve up to k straight line movements. The option of performing k-step removals, for some constant $k > 2$ substantially increases the lower bound on the portion of removable vertices. We show that when removing a vertex v with degree q, then k is bounded from above by either $q-4$, or by the number of concave (reflex, i.e. $> \pi$) corners on the link of v (see Section 3 for a definition), and from below by $\lfloor q/3 \rfloor$.

In Section 2 we define the k-step remove operation, and in Section 3 we prove an upper bound related to k, as well as a lower bound. In Section 4 we show that a hierarchical graph of size $O(n)$ and depth $O(\log(n/m))$ can be achieved using 2-step removals in linear time, while maintaining m important vertices. In Section 5 we give additional arguments for the usefulness of the 2-step removals, by showing that they increase in number as the number of 1-step removable vertices decreases. Their exact relationship is determined. Finally, we give a brief overview of future research in Section 6.

For simplicity, in the text we always assume, unless mentioned otherwise, that we have no three collinear points, and hence we do not have to consider angles with degree 180. Our results also hold if we allow collinear points and in this case treat angles of 180 degrees as concave.

2. Basic definitions

In this section we define some basic operations and notations. As input we are given a triangulation $T = (V, E)$ with a simple polygon P as boundary, and a set $U \subseteq V$ of m important vertices, where V is the set of n vertices of the triangulation and E is the set of edges, see Fig. 2. We let P_e denote the edges of P, and we let P_v denote the vertices of P, where $P_v \subseteq U$, meaning that neither P_e or P_v will be removed during the simplification. We call the vertices in P_v *exterior* vertices and the rest *interior* vertices. Next, let $N_T(v) := \{w \in V | (v, w) \in E\}$ denote the neighboring vertices of a vertex $v \in V$. A *restricted edge contraction*, see Fig. 1, of an edge $e = (u, v) \in E$ *on (or onto)* v, is an operation that removes u, v and for each edge (u, w), if $w \neq v$, and $(v, w) \notin E$, then it replaces (u, w) by (v, w) in T, and if $w \neq v$, and $(v, w) \in E$, then (u, w) is removed from T. The inverse operation where vertex v is split into two vertices $u \in V$ and $v \in V$ is called a *vertex split*, see Fig. 1. In this paper we only consider such restricted edge contractions. Furthermore, for the remainder we say edge contraction as short for restricted edge contraction. In a graphics related context, an edge contraction can be shown smoothly, by continuously displaying a straight motion of u to v, while the edges adjacent to u are maintained as straight connections to u. Moreover, the aim is to simplify T by iteratively performing edge contractions.

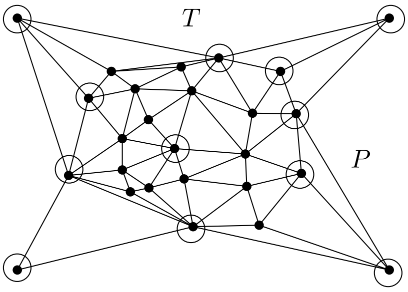
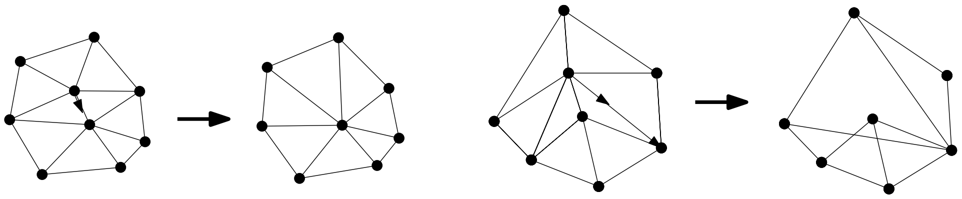


Fig. 2. A planar triangulation T, with a simple polygon P as boundary, is given as input, as well as m important vertices (circled) that may not be removed.

A problem that often occurs during edge contractions of a triangulation is that the resulting graph might not be a planar triangulation. An edge contraction is said to be *valid* if the resulting graph is still a planar triangulation (see Fig. 3a), and *invalid* otherwise (see Fig. 3b).

Definition 1. Given an interior vertex $v \in V$, the link of v, denoted $link(v)$, is the cycle of T passing through the neighbors of v, where the edges of the cycle form the boundary of the union of the triangles incident to v. Furthermore, let $I(v)$ denote the closed region bounded by $link(v)$. We say that two vertices $u, u' \in N_T(v) \cup \{v\}$ *see* each other if the straight-line segment between u and u' lies entirely within $I(v)$.

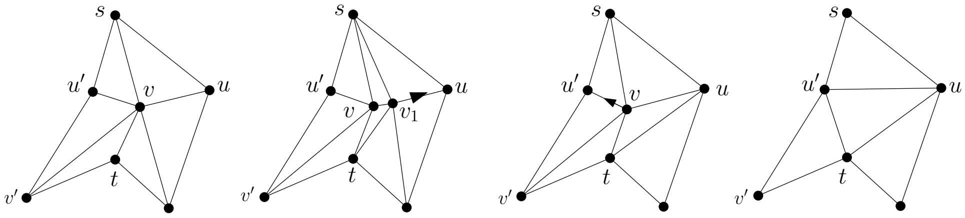


(a) (b)

Fig. 3. (a) A valid edge contraction. (b) An invalid edge contraction.

We consider a generalized edge contraction. For this purpose we first define a *split-and-contract* operation.

Definition 2. Given a vertex $v \in V$ and vertices $s, t, u \in link(v)$, let $C(s, t, u)$ be the vertices (s and t included) of the chain of $link(v)$ which connects s and t, and includes u. A split-and-contract from v to u, using s and t (illustrated in Fig. 4a-c) denotes an operation where v is split into two vertices, v and v_1, such that v_1 is connected to $C(s, t, u) \cup \{v\}$, and v is connected to $\{N_T(v) \setminus C(s, t, u)\} \cup \{s, t, v_1\}$. After this split the edge (v_1, u) is contracted on u. The split-and-contract operation is said to be *valid* if the triangulation is planar at every step of the operation.



(a) (b) (c) (d)

Fig. 4. Illustrating a 2-step contraction of a degree 6 node v.

Note that a split-and-contract does not reduce the size of T. However, when an edge is contracted, two vertices are replaced by one. Thus, we define the concept of a 1-*step removable* vertex and generalize this concept into a k-*step removable* vertex.

Definition 3. If there exists an edge $e = (u', v) \in E$ such that e is validly contractible on v, then we say that u' is 1-*step removable on (or onto)* v. Such an operation is called a 1-*step removal.*

Definition 4. A vertex $v \in V$ is said to be k-step removable if there exist vertices $s, t, u \in link(v)$ such that a valid split-and-contract from v to u, using s and t can be made, and after this split-and-contract, v is $(k-1)$-step removable. Such an operation is called a *k-step removal*.

It is clear that a k-step removable vertex is also $(k+1)$-step removable. Figures 4a-d show a vertex v that is 2-step removable since a valid split-and-contract from v to u, using s and t is followed by a 1-step removal of v on u'.

3. Characterization of *k*-step removals

In this section we consider k-step removals, and show bounds (Theorem 1, Lemma 2 and Theorem 3) related to k and the removability of a vertex v. We start with the lower bounds.

Lemma 1. *In order to remove an interior vertex $v \in V$ with degree at least six a 2-step removal may be required.*

Proof. Figure 5a illustrates an example where no vertex on $link(v)$ can see all other vertices in $link(v)$, hence the lemma follows. □

For a more general lower bound we consider a vertex v as shown in Fig. 5b. The vertex is such that $link(v)$ has a sunlike shape, where a ray is such that its endpoint u is seen only by its two neighbors u' and u'', and v.

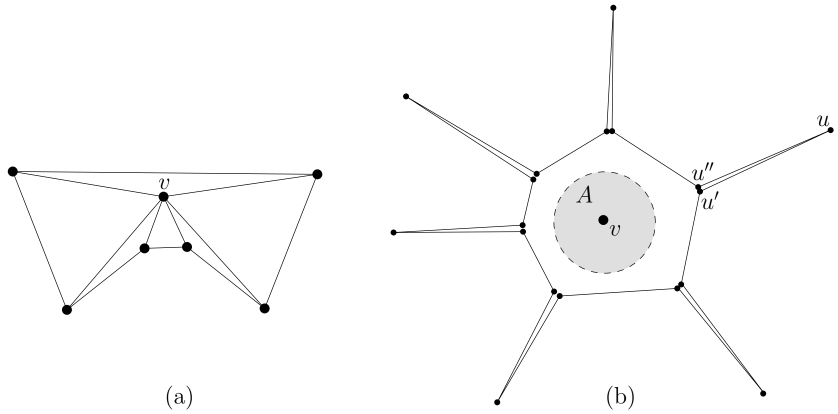


Fig. 5. (a) Example of a vertex v that is not 1-step removable. (b) The structure used to prove the lower bound.

Lemma 2. *In order to remove an interior vertex $v \in V$ of degree $q > 5$, a $(\lfloor q/3 \rfloor)$-step removal may be required.*

Proof. Using the setting illustrated in Fig. 5b it holds that as long as at least two rays remain on $link(v)$ no 1-step removal of v is possible. Further, in order to perform a split-and-contract operation we need vertices r, s and t such that r sees all vertices on the chain between s and t in which r is included. Thus, since every ray endpoint u is visible only from its two neighbors u' and u'' we have that as long as at least two rays remain at most one ray may be removed from $link(v)$ as the result of a split-and-contract. As the structure contains at least $\lfloor q/3 \rfloor$ rays the lemma follows. $\square$

With regards to guaranteeing a hierarchical simplification graph of small size and depth, we mainly consider 2-step removals (Section 4). However, depending on complexity, some areas of an object may require more triangles than others, such as, for example, the nose of a face, versus the more flat cheek. Thus, it would be desirable to be able to choose local areas in which to remove vertices. However, as Lemma 2 shows, 2-step removals may not always be sufficient for a specific area. Consider, for example, the area marked as A in Fig. 5b. This area contains only one vertex v of degree $q \geq 9$ (a $(\lfloor q/3 \rfloor)$-step removal may be required). Thus, a generalized k-step removal is needed.

Next we focus on finding an upper bound. Assume without loss of generality that v has c concave vertices on $link(v)$, as shown in Fig. 10a. Let s_1 be a concave vertex farthest from v and order the concave vertices $s_1, \ldots s_c$ as they appear clockwise around v (in this context, let $i+1 = 1$ if $i = c$, and let $i-1 = c$ if $i = 1$). Next, let β_i denote the angle $\angle s_{i-1}s_is_{i+1}$ and let α_i denote the angle $\angle s_ivs_{i+1}$. Moreover, let $C(s_i)$ denote the subchain of $link(v)$ clockwise from s_i to s_{i+1} and let $C_P(s_i)$ denote the convex polygon bounded by $C(s_i)$ and the edge (s_i, s_{i+1}). The following lemma will be needed:

Lemma 3. *If* $\alpha_i \leq 180°$ *then the two consecutive concave vertices* s_i *and* s_{i+1} *must see each other.*

Proof. Since s_i and s_{i+1} are consecutive concave vertices of $link(v)$, the chain $C(s_i)$ and the vertex v must lie on opposite sides of the line s_is_{i+1}, and the lemma follows. $\square$

Theorem 1. *Every interior vertex* v *with degree at most* k *is* q*-step removable, where* $q = \max\{1, k-4\}$.

Proof. The theorem is proven by induction on the degree of v.

Base case: Vertices of degree at most four can easily be shown to be 1-step removable. We thus assume that v has degree five, as shown in Fig. 6. A vertex q is concave/convex, if the angle of the two edges incident on q is concave/convex. If there exists a vertex v' of $link(v)$ which can see all other vertices of $link(v)$ then v is 1-step removable on v', and the theorem holds. Below we prove that $link(v)$ will always contain at least one vertex that can see all the other vertices of $link(v)$.

If $link(v)$ is a convex polygon then every vertex can see all the other vertices. If $link(v)$ has one concave vertex v' then v' can see all other vertices of $link(v)$, see

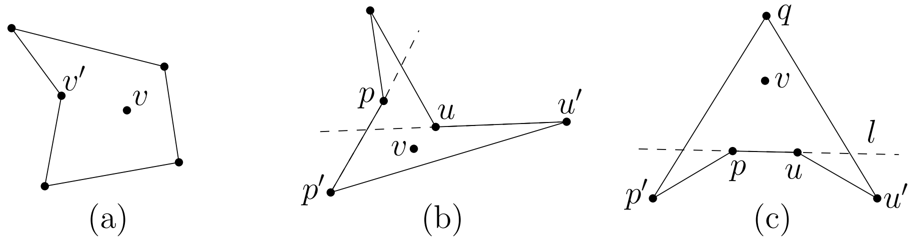


Fig. 6. (a) A degree five vertex v with only one concave vertex in $link(v)$. (b) and (c) illustrates the two cases occurring in the proof of Theorem 1. Edges between v and $link(v)$ are not included in order to avoid cluttering of the figure.

Fig. 6a. Otherwise, $link(v)$ has two concave vertices (it cannot have three), and we have two cases as illustrated in Fig. 6b and 6c, respectively. In the first case the concave vertices p and u are not adjacent, while in the second they are.

First case: Note that p and u must lie inside the triangle defined by the three vertices of the convex vertices in $link(v)$, and that p and u always see each other. There exist two adjacent convex vertices p' and u', such that p' is adjacent to p and u' is adjacent to u. It is straightforward to see that p sees all vertices of $link(v)$ if the edge (p, p') does not intersect the extension ray of the edge (u, u'), as p then can see u'. The same holds for u, the edge (u, u') and the line extension of (p, p'). However, both cases cannot occur simultaneously as this implies that the edges (u, u') and (p, p') must cross. Thus, either p or u can see all of $link(v)$.

Second case: Consider the one convex vertex q not adjacent to either p or u. Let p', u' be the other two vertices such that p' and u' are adjacent to p and u, respectively. Since q is adjacent to both p' and u', q will see all vertices of $link(v)$ if and only if q sees both p and u. Next, consider the line l through p and u. Since p and u are concave vertices, p' and u' must lie on the same side of l. Furthermore, q must connect to p' and u' such that p' and u' are convex and p and u are concave. This means that q must lie on the opposite side of l with respect to p' and u', which immediately implies that q can see both p and u.

Thus, v is 1-step removable in both subcases and the base case holds.

Induction hypothesis: Assume that the theorem holds for all vertices of degree at most $m - 1$.

Induction step: Assume that v has degree m. We show that there always exists a valid split-and-contract, where the result of a split-and-contract is that one edge incident to v is flipped.

If $link(v)$ contains at most one concave vertex then v is 1-step removable as there exists a vertex in $link(v)$ that sees all other vertices. Thus, we may assume that $link(v)$ contains at least two concave vertices. Since v sees all of $link(v)$, there exists at least one s_i such that $v \notin C_P(s_i) \setminus (s_i, s_{i+1})$. Consider a vertex $u_1 \in C(s_i) \setminus \{s_i, s_{i+1}\}$ and its two neighbors $u_2, u_3 \in C(s_i)$, as well as the neighbor

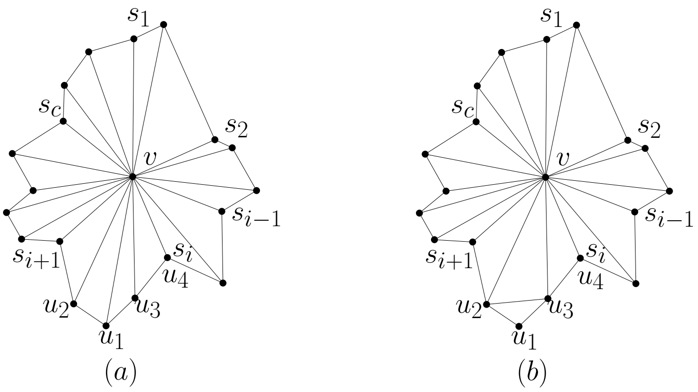


Fig. 7. (a) A split-and-contract from v to u_3, using u_2 and u_4 is valid since u_3 sees u_1, u_2 and u_4 (b) The split-and-contract results in the edge (v, u_1) being flipped.

$u_4 \in link(v)$ of u_3 as shown in Fig. 7a. Since v sees all of $link(v)$ we have that $u_3 \in C(s_i)$ must see u_1, u_2 and u_4. Thus, we can perform a split-and-contract from v on u_3, using u_2 and u_4, resulting in the edge (v, u_1) being flipped to (u_2, u_3), see Fig. 7b. Thus, the degree of v is now $m - 1$, and applying the induction hypothesis on v proves the theorem. □

Next we show a theorem that follows from Theorem 1. First we need two simple observations. Recall that P_v is the set of vertices on the boundary of T.

Observation 1. If $|P_v| = 3$, and the number of interior vertices is at least one, then each vertex in P_v must have degree at least three, see Fig. 8a.

Observation 2. Any triangulation of P has $(2n-2-|P_v|)$ triangles and $(3n-3-|P_v|)$ edges.

This last observation follows from the proof of Theorem 9.1 in the book by de Berg et al. [5], which immediately implies that the total degree of T is $(6n-6-2|P_v|)$.

Theorem 2. *If the simple polygon P, on the boundary of the triangulation T contains at most five vertices, the number of vertices of T is strictly greater than the number of vertices of P, and all vertices not in P are non-important, then there exists at least one 1-step removable vertex in T.*

Proof. We have three cases:

Case 1: $|P_v| = 3$. From Observation 1 we know that all vertices in P_v must have degree at least three, thus at least nine in total. We know that the total degree of all interior vertices is at most $6n - 6 - 2|P_v| - 9 = 6n - 21$. Next, from Theorem 1 we know that a vertex of degree at most five is 1-step contractible. Since no vertex in P_v is removed, this means that in the worst case all vertices in P_v have degree exactly three, while a maximum number of interior vertices have degree at least six. However, not all interior vertices can have degree at lest six, since the total degree of all interior vertices would then be at least $6(n-3) = 6n-18$, which is a contradiction. Thus, there exists at least one interior vertex v of degree at most five, which means that v is 1-step removable.

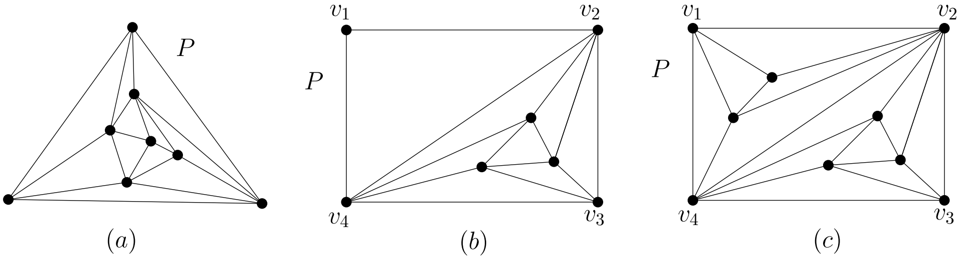


Fig. 8. (a) If $|P_v| = 3$ then all vertices P_v must have degree at least three. (b) First subcase for $|P_v| = 4$, where vertex v_1 has degree two. (c) The second subcase for $|P_v| = 4$, where all vertices in P_v have degree at least three.

Case 2: $|P_v| = 4$. Let $v_1, \dots, v_4$ be the vertices in P_v in clockwise order. We have two subcases. The first is that one of the vertices in P_v has degree two, see Fig. 8b. If v_1 has degree two then both of its neighbor endpoints v_2 and v_4 must be connected with an edge in order to form a triangle. Further, this triangle must be empty, thus, all interior vertices of T must lie in the triangle v_2, v_3, v_4. This triangle must contain at least one 1-step removable vertex as shown in Case 1 above. The second subcase is that no vertex in P_v has degree two, see Fig. 8c. In this case the total degree of all vertices on P_v must be at least twelve. Thus, the total degree of all interior vertices is at most $6n - 6 - 2|P_v| - 12 = 6n - 26$.

Not all interior vertices can have degree at least six, since the total degree of all interior vertices would then be at least $6(n-4) = 6n - 24 > 6n - 26$, which is a contradiction. Thus, there must exist at least one interior vertex v of degree at most five, which means that v is 1-step removable.

Case 3: $|P_v| = 5$. Let $v_1, \dots, v_5$ be the five vertices of P_v in clockwise order. We have two subcases. If one of the vertices, say $v_1 \in P_v$, has degree two then both of its neighbor endpoints $v_2, v_5 \in P_v$ must be connected with an edge in order to form a triangle. Further, this triangle must be empty, and thus, all interior vertices of T must lie in the quadrangle $v_2, v_3, v_4, v_5 \in P_v$. This quadrangle must contain at least one 1-step removable vertex as shown in Case 2 above. The second subcase is that no vertex in P_v has degree two, see Fig. 9b. In this case the total degree of all vertices on P_v must be at least fifteen. From Observation 2 we know that the total degree of T in this case is $6n - 6 - 2|P_v| = 6n - 16$, which means that the total degree of all interior vertices is at most $6n - 16 - 15 = 6n - 31$.

Finally, not all interior vertices can have degree at least six, since the total degree

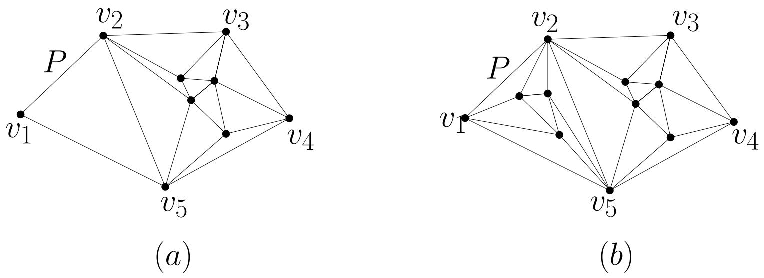


Fig. 9. (a) First subcase for $|P_v| = 5$, where vertex v_1 has degree two. (b) The second subcase for $|P_v| = 5$, where all vertices in P_v have degree at least three.

of all interior vertices would then be at least $6(n-5) = 6n-30 > 6n-31$, which is a contradiction. Thus, there must exist at least one interior vertex v of degree at most five, which means that v is 1-step removable. □

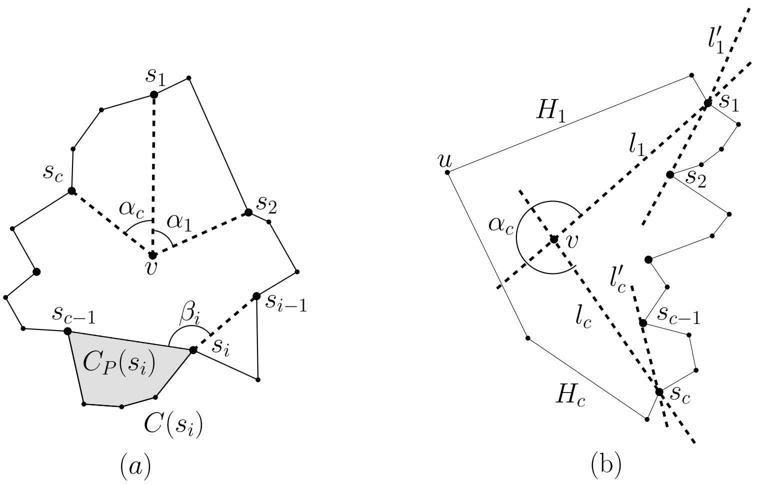
From the theorem it immediately follows that if a region bounded by a non-empty cycle of length smaller than six in T contains no important vertices, then there exists at least one 1-step removable vertex in T. Further, the above theorem is in a sense tight since Lemma 1 tells us that if P_v contains at least six vertices, it may be that no interior vertex of T is 1-step removable. One should also note that if only a few vertices are 1-step removable then almost all interior vertices in T must have degree 6, while simultaneously being not 1-step removable. However, we have not been able to construct any such examples, thus the bound stated in Lemma 2 might be very conservative.

Next, we present an upper bound on k, such that every vertex in T is k-step removable. As only concave corners restrict visibility, intuitively it should be easier to remove a vertex with few concave corners on its link. A valid split-and-contract which reduces the number of concave corners by at least one in the resulting $link(v)$ is denoted a *reducing* split-and-contract.

Lemma 4. *If $\beta_i < 180°$, $v \notin C_P(s_{i-1})$ and $v \notin C_P(s_i)$, then a split-and-contract from v to s_i, using s_{i-1} and s_{i+1}, is reducing.*

Proof. Since $v \notin C_P(s_{i-1})$ and $v \notin C_P(s_i)$ it immediately follows that $\alpha_{i-1} < 180°$ and $\alpha_i < 180°$, which means (Lemma 3) that s_{i-1} and s_i see each other, as do s_i and s_{i+1}. This makes the split-and-contract valid since all the vertices of $C(s_{i-1})$ and $C(s_i)$ see s_i, to which they will be connected after the split-and-contract. Further, since $\beta_i < 180°$, s_i will be convex after the split-and-contract and no new concave corners can appear on the resulting $link(v)$, and the lemma follows. □

The following theorem can now be shown (see Fig. 10 for an illustration). Recall that s_1 is the concave vertex farthest from v and that the concave vertices are ordered $s_1, \ldots s_c$ as they appear clockwise around v.



(a) (b)

Fig. 10. An illustration of the two cases in the proof of Theorem 3.

Theorem 3. *Every interior vertex $v \in V$ with at most c concave vertices on its link is c-step removable.*

Proof. The theorem is proven by induction on c.
Base cases: If $c = 1$ then let s be the concave vertex on $link(v)$. As s can see all other vertices on $link(v)$, v can be 1-step removable on s, thus the theorem holds for $c = 1$.
Induction hypothesis: Assume that the theorem holds when $link(v)$ contains at most $c-1$ concave vertices.
Induction step: Assume $link(v)$ contains c concave vertices. We show that there always exists a reducing split-and-contract, and thus, the theorem is proved by applying the induction hypothesis. Consider the following cases:

Case 1: $\alpha_c < 180°$ and $\alpha_1 < 180°$. In this case $c \geq 3$ otherwise either $\alpha_1 \geq 180°$ or $\alpha_c \geq 180°$. We also immediately have that $v \notin C_P(s_c)$ and $v \notin C_P(s_1)$ since both α_c and α_1 have degree strictly smaller than 180°. Furthermore, since s_1 is a vertex in $link(v)$ furthest from v, it holds that $\beta_1 < 180°$. Thus, a split-and-contract from v to s_1, using s_c and s_2, is reducing according to Lemma 4.

Case 2: $\alpha_c \geq 180°$ or $\alpha_1 \geq 180°$. Assume without loss of generality that $\alpha_c \geq 180°$ which immediately implies that $v \in C_P(s_c)$, see Fig. 10b. Since s_1 and s_c both are visible from v there can be no other vertex s_i, $i \neq c$, such that $v \in C_P(s_i)$ (except for the case $c = 2$, where we have that $v \in C_P(s_1)$ and $v \in C_P(s_c)$ when $\alpha_1 = \alpha_c = 180°$. However, this does not change the validity of the proof below). Let l_1 and l'_1 be the lines through s_1 and v, and through s_1 and s_2, respectively. Let l_c and l'_c be the lines through s_c and v, and through s_c and s_{c-1}, respectively. Line l_1 defines two halfplanes, where we consider the halfplane containing none of the vertices

$s_2, s_3, \ldots s_c$ (s_1 is on the border of this halfplane). Also, l'_1 defines two halfplanes, where we consider the one not containing $C_P(s_1)$. Let H_1 be the intersection of these two halfplanes, as shown in Fig. 10b. H_c is defined correspondingly using lines l_c and l'_c. Assume that H_1 contains a vertex $u \in C(s_c)$, and consider a split-and-contract from v to s_1, using u and s_2. All vertices between u and s_1 are visible to s_1, since $H_1 \cup H_c$ can contain vertices from $C(s_c)$ only. The same holds between s_1 and s_2 since $v \notin C_P(s_1)$. Moreover, the definition of l_1 guarantees that u will remain in the resulting $link(v)$ after the split-and-contract, and line l'_1 guarantees that the corner defined by edges (u, s_1) and (s_1, s_2) is convex. Thus, if H_1 contains a vertex $u \in C(s_c)$, the above split-and-contract must be reducing. Correspondingly, a split-and-contract from v to s_c, using u and s_{c-1} will be reducing if H_c contains a vertex $u \in C(s_c)$. Finally, the area $H_1 \cup H_c$ must contain a vertex from $C(s_c)$ since $\alpha_c > 180$ and $C(s_c)$ connects s_c and s_1. □

3.1. *The number of k-step removable vertices*

In this section we show a linear bound on the number of k-step removable vertices. Throughout the section we assume that $|P_v| = 4$ and that $m = 0$. The results for this restricted case will then be used in Section 4 to achieve more general results.

The first bound follows from Theorem 1 and the fact that the total degree is bounded.

Lemma 5. *If $|P_v| = 4$ and $m = 0$ then at least $(\frac{k-1}{k+2})(n-4)$ interior vertices are k-step removable, for every $k \geq 2$.*

Proof. Let L be the set of interior vertices of T that are k-step removable. From Theorem 1 we know that L contains all vertices of degree at most $d = k + 4$. Let N be the remaining set of interior vertices. Recall that the sum of the degrees over all vertices is exactly $6n - 6 - 2|P_v| = 6n - 14$. We also have that the total degree of P_v must be at least 12. This holds since a vertex $v \in P_v$ with degree two implies that its two neighbors $v', v'' \in P_v$ must have degree at least four in order to guarantee a planar triangulation. Moreover, as the vertices in N have degree at least $d + 1$ and the vertices in L have degree at least three, we have the following equation:

$$12 + (d+1)|N| + 3|L| \leq 6n - 14 \text{ where } |N| + |L| = n - 4.$$

As a result it holds that $|L| \geq \left(\frac{d-5}{d-2}\right)(n-4) \geq \left(\frac{k-1}{k+2}\right)(n-4)$. □

A similar bound on the number of k-step removable vertices as in Lemma 5 can also be derived without using sums of degrees, and by using Theorem 3 instead [2]. This approach turns out to be more complicated, requiring a longer proof, and the bound it achieves is not significantly better than the one stated in Lemma 5. Hence, we do not include it in this journal version of the paper, and we refer the interested reader to [2].

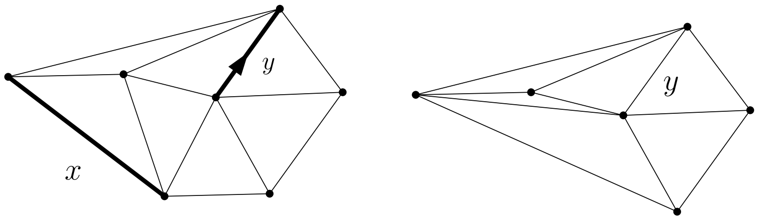


Fig. 11. Initially edges x and y are contractible. After the contraction of x, y is no longer contractible.

4. The hierarchical graph

Next we create the *level-of-detail hierarchy*, denoted L, of T where the general procedure is as follows. We start with T, which we denote the first *level*. A constant fraction of the vertices are then simultaneously removed, in a *round* of k-step removals, resulting in a triangulation which we denote the second level. Correspondingly, level i of L is created through one round of k-step removals from level $i-1$ in L. Next, we represent L with a directed, acyclic graph H, denoted a *hierarchical graph*. A level i in L is a triangulation, while the corresponding level i in H is a set of nodes, where each triangle in level i of L is represented with a node in level i of H. Furthermore, one way to view a k-step removal of a vertex v is as removing v and all edges adjacent to v, leaving $I(v)$ empty, and then re-triangulating $I(v)$. Thus, we place a directed edge between a node u at level $i-1$ in H, and a node u' at level i of H, if both the triangle corresponding to u disappeared, and the triangle corresponding to u' appeared, as a result of the same k-step removal of a vertex v. We let U denote the set of nodes, and F denote the set of edges in H. Moreover, we say that the sum $|U| + |F|$ is the *size* of H, and that the number of levels of H is the *depth* of H.

In order to guarantee a hierarchical graph of small size and depth, several vertices must be simultaneously removed in each round. Ideally, these vertices would be 1-step removable. However, currently we are only able to guarantee at most one 1-step removable vertex, as seen in Lemma 2. Therefore we consider 2-step removable vertices. We show that using 2-step removals, given m *important* vertices or edges, we can achieve a hierarchical graph of size $O(n)$ and depth $O(\log(n/m))$, such that the simplified resulting triangulation contains the m important vertices and edges and at most $O(m)$ other vertices from T. Note that a previously valid contractible edge might become invalid after other edges have been contracted, as shown in Fig. 11. In order to avoid this problem, for the purpose of finding simultaneously removable vertices, we consider non-adjacent vertices, that is *independent* vertices.

Theorem 4. *Given a triangulation $T = (V, E)$ with a simple polygon P as boundary, and m important (the vertices of P included) vertices $U \subset V$, one can perform $O(\log(n/m))$ rounds of 2-step removals to obtain a triangulation T' of a vertex* *set* V' *with complexity* $O(m)$ *such that* $U \subseteq V' \subset V$*. All rounds can be performed in* $O(n)$ *total time, and such that the corresponding hierarchical graph has size* $O(n)$ *and depth* $O(\log(n/m))$*.*

Proof. Assume that a rectangle R is added, see Fig. 12, such that T is included in R, and that edges are added such that the interior of R becomes triangulated. Denote this triangulation $Q = (S, D)$, where $n' = |S| = n + 4$, $m' = m + 4$, and the vertices of R are set as important.

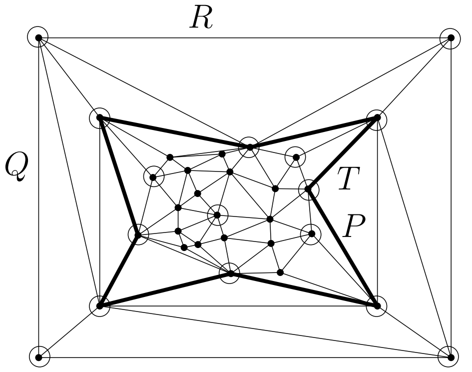


Fig. 12. A rectangle R is added such that T is included in R. The vertices of R are set as important, and edges are added to the interior of R so that a triangulation Q is achieved.

Let S_2 be the set of 2-step removable interior vertices of Q of degree at most six, assuming that $m' = 0$. Lemma 5 was shown using Theorem 1, thus, we have that $|S_2| \geq \frac{n'-4}{4} \geq \frac{n'}{20}$, $n' \geq 5$. Since a vertex in S_2 has at most six neighbors we can choose at least $\frac{n'}{20 \cdot 7} = \frac{n'}{140}$ vertices from S_2 such that none of the chosen vertices has a neighbor from S_2. Thus, there exists a constant fraction $\gamma \geq \frac{1}{140}$ of independent 2-step removable vertices from the n' vertices and, hence, there are $n'/140 - m'$ vertices for removal.

Let n'_i denote the number of vertices before round i and consider an arbitrary constant $\delta < \gamma$. Perform rounds on Q until $m' \geq \delta n'_i$, that is until the resulting vertex set S' has complexity $O(m')$. This is possible, since as long as $m' \leq \delta n'_i$, there are at least $\gamma n'_i - \delta n'_i = (\gamma - \delta) n'_i$ 2-step removable vertices remaining, containing no important vertex. Thus, a triangulation Q' can be obtained using at most $O((\log_{\frac{1}{1-(\gamma-\delta)}} 2) \cdot (\log n' - \log m')) = O(\log(n'/m'))$ rounds of removals. Since $m = O(m')$ and $n = O(n')$ this immediately implies that one can perform $O(\log(n/m))$ rounds of 2-step removals on T to obtain a triangulation T' of a vertex set V' with complexity $O(m)$ such that $U \subseteq V' \subset V$. Hence, the construction of a hierarchical graph of depth $O(\log(n/m))$ is possible.

Now, we estimate the size of hierarchical graph. The number of nodes in the hierarchical graph is, since $n_i \leq n\gamma^{i-1}$, at most $O(n + n\gamma + n\gamma^2 + \ldots + n\gamma^{O(\log(n/m))}) =$

$O(n)$. Further, only 2-step removable vertices of degree at most six are used during the rounds of removals. This means that at most six triangles are removed and at most five triangles added as a result of the 2-step removal, which implies that each node in the hierarchical graph has at most five incident outgoing edges. Thus, in total the number of edges of the hierarchical graph is $O(n)$, which also means that the size is $O(n)$.

Finally we consider the time complexity of creating the hierarchical graph. Theorem 4 was shown using only 2-step removable vertices of constant degree (at most six). Thus, in each round i the set of γn_i independent 2-step removable vertices can be found in $O(n_i)$ time. Again, since $n_i \le n\gamma^{i-1}$ the total running time is $O(n + n\gamma + n\gamma^2 + \ldots + n\gamma^{O(\log(n/m))}) = O(n)$. □

The above results also hold for m important edges (or m edges and vertices, in total), since each important edge restricts possible removals for only a constant number of vertices.

5. The relationship between 1-step and 2-step removable vertices

In this section we show a relationship between the number of 1-step removable vertices and the number of 2-step removable vertices, assuming that no interior vertex is important. It follows from these relations that the lower bound on the number of 2-step removable vertices increases as the number of 1-step removable vertices decreases. A motivation for studying these bounds is that although it would be desirable to have a larger number of 1-step removable vertices, we show that a lack of such vertices is to a certain degree compensated by a guarantee that there is a larger portion of 2-step removable vertices. This serves also as a further motivation for considering 2-step removals.

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ denote the set of 1-step and 2-step removable interior vertices, respectively. We say that a vertex v *consumes* x concave corners if $link(v)$ contains x concave corners. Moreover, for simplicity we assume that the total degree over all vertices is $6n$. This is done in order to simplify the calculations. Note, however, that increasing the total degree decreases $|\mathcal{T}_2|$ as the number of vertices of degree at least seven increases. Thus, the lower bounds are still valid.

Lemma 6. *If* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *are the sets of* 1*-step and* 2*-step removable interior vertices, respectively, of the input triangulation, and* P_v *is the set of vertices on the boundary of the input triangulation, then*

$$|\mathcal{T}_1| \le x(n - |P_v|) \Rightarrow |\mathcal{T}_2| \ge \frac{n}{3}(1-x) - 3|P_v|, 0 \le x \le 1$$

Proof. Only vertices of degree three can generate three concave corners, and from Theorem 1 we know that only 1-step removable vertices can have degree three. Also, in the worst case all vertices in P_v have degree three. Thus we have at most

$$3(x(n - |P_v|) + |P_v|) + 2(n - (x(n - |P_v|) + |P_v|)) \le n(2 + x) + 5|P_v|$$

concave corners which means that at most $\frac{n(2+x)+5|P_v|}{3} \leq \frac{n(2+x)}{3} + 2|P_v|$ vertices can consume at least three concave corners, thus not being 2-step removable (Theorem 3). In the worst case all these vertices are interior, and as a result at least

$$n - |P_v| - \frac{n(2+x)}{3} - 2|P_v| = \frac{n}{3}(1-x) - 3|P_v|$$

are 2-step removable. □

The above bound can be slightly improved, as shown in the following Theorem:

Theorem 5. *If $\mathcal{T}_1$ and $\mathcal{T}_2$ are the sets of 1-step and 2-step removable interior vertices, respectively, of the input triangulation, and P_v is the set of vertices on the boundary of the input triangulation, then*

$$|\mathcal{T}_1| \leq x(n - |P_v|) \Rightarrow |\mathcal{T}_2| \geq n(1-3x) - 3|P_v|, \quad \text{for} \quad 0 \leq x \leq 1$$

Proof. Since 1-step removable vertices must have degree at least three we have at most $3(x(n - |P_v|) + |P_v|)$ vertices of degree at least seven. This follows since a vertex of degree three 'allows' three vertices of degree at least seven. Finally, we have at least $n - 3(x(n - |P_v|) + |P_v|) = n(1-3x) - 3|P_v|$ vertices of degree six (which are 2-step removable). □

Both results 'fail' for $x \geq 1/4$ as the lower bound of $\mathcal{T}_2$ becomes smaller than the upper bound of $\mathcal{T}_1$. Further, for $x \leq 1/4$ the lower bound for $\mathcal{T}_2$ is better for Lemma 5, making this lemma stronger than Lemma 6.

We may also consider whether Theorem 5 can be refined using Theorem 3. Such a refinement would be possible if not all vertices of degree at least seven were able to consume at least three concave corners. However, since

$$3 \cdot 3(x(n - |P_v|) + |P_v|) \leq n(2+x) + 5|P_v|$$

for $x \leq \frac{2n}{8n - |P_v|}$, this does not hold (unless we are able to improve bounds, such as, for example, the total number of generated concave corners).

6. Further research

Currently a number of possible extensions of the above results are being explored.

- To what extent do the results generalize to three dimensions?
- Is it possible to maintain bounded degree for all vertices during rounds of removals?
- Can the user specify to a greater degree which edges will be contained in certain levels of the hierarchical graph (maybe even including edges not in the original graph)?
- Can we obtain better upper and lower bounds for the number of 1-step removable vertices?
- Would implementing the algorithm and testing it on real data confirm an efficient performance and behavior?

7. Acknowledgement

We thank Tomas Akenine-Möller for valuable discussions and comments.

References

1. G. Aloupis, P. Bose and P. Morin. Reconfiguring Triangulations with Point Moves and Edge Flips. Algorithmica, 47(4):367–378, 2007.
2. M. Andersson. Approximation Algorithms for Geometric Networks. Ph.D. thesis, Lund University, Department of Computer Science, Dissertation 23, 2007.
3. P. Bose, J. Czyzowicz, P. Morin, J. Gao and D. Wood. Simultaneous Diagonal Flips in Plane Triangulations. Journal of Graph Theory, 54(4):307–330, 2007.
4. S. Cheng, T. Dey and S. Poon. Hierarchy of Surface Models and Irreducible Triangulations. Computational Geometry Theory & Applications, 27(2):135-150, 2004.
5. M. de Berg, M. van Kreveld, M. Overmars, and O. Schwarzkopf. Computational Geometry: Algorithms and Applications. Springer-Verlag, Berlin, Germany, 2nd edition, 2000.
6. M. de Berg, K. T. G. Magillo. On Levels of Detail in Terrains. In Proc. *11th ACM Symposium on Computational Geometry*, ACM Press, pp.26-27, 1995.
7. L. De Floriani, P. Magillo and E. Puppo. Building and Traversing a Surface at Variable Resolution. In Proc. *8th IEEE Conference on Visualization*, pp. 103-110, 1997.
8. C. A. Duncan, M. T. Goodrich and S. G. Kobourov. Planarity-Preserving Clustering and Embedding for Large Planar Graphs. Computational Geometry Theory & Applications, 24(2):95–114, 2003.
9. M. Garland. Multiresolution Modelling: Survey and Future Opportunities. Eurographics, State of the Art Report (STAR), pp. 111-131, 1999.
10. S. Gumhold, P. Borodin and R. Klein. Intersection Free Simplification. In Proc. *4th Israel-Korea Bi-National Conference on Geometric Modeling and Computer Graphics*, pp 11-16, 2003
11. P. Heckbert and M. Garland. Surface Simplification Using Quadric Error Metrics. In Proc. *24th Conference on Computer graphics and interactive techniques*, pp. 209-216, 1997.
12. P. Heckbert and M. Garland. Survey of Polygonal Surface Simplification Algorithms. SIGGRAPH Course Notes, no. 25, ACM Press, New York, 1997.
13. H. Hoppe. Progressive Meshes. In Proc. In Proc. *23rd Conference on Computer graphics and interactive techniques*, pp. 99-108, 1996
14. D. G. Kirkpatrick. Optimal Search in Planar Subdivisions. SIAM Journal of Computing, 12:28-35, 1983.
15. J. C. Xia, J. El-Sana and A. Varshney. Dynamic View-Dependent Simplification for Polygonal Models. In Proc. *7th IEEE Conference on Visualization*, pp.327-334, 1996.