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COMP5045 Computational Geometry

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COMP5045 – Computational Geometry

Course page:
http://www.gudmundsson.biz/comp5045/

Lecturers:
Joachim Gudmundsson & Thomas Wolle
NICTA, School of IT Building, Level 5 West

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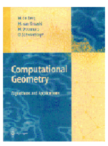
Course book:
M. de Berg, M. Overmars, M. van Kreveld and O. Schwarzkopf
Computational Geometry: Algorithms and Application, Springer-Verlag



Outline:
13 lectures, Thursday 14:00-16:00

Assignments:
3 assignments, emailed as pdf-file (preferably using latex)
Handing in late is not accepted!

Grade:
Each assignment is 25% of your grade
Final exam is 25% of your grade (you must not fail the exam)

Collaboration:
General ideas - Yes!
Formulation and writing - No!

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Brief History: Algorithms and CG

Algorithms:
300 BC: Euclid designed an algorithm for GCD.
780 AD: al-Khowarizmi – the word "Algorithm" is derived from his name.
…
Babbage, Cantor, Hilbert, Church, Gödel
…
1936: Turing developed a machine that provides a formal and rigorous definition of an algorithm

Computational Geometry:
1644: Descartes wrote about Voronoi diagrams
1759: Euler & Vandermonde discussed Euclidean TSP
1978: Shamos wrote his thesis which defines the start of modern CG
1985: Preparata & Shamos wrote the first CG textbook

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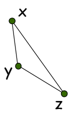
What is computational geometry?

The study of algorithms to solve problems stated in terms of geometry.

The problems we study are defined in a metric space!

For every two points x and y in the metric space, there is a function $d(x,y) \geq 0$ which gives the distance between them as a nonnegative real number. A metric space must also satisfy

1. $d(x,y) = 0$ iff $x = y$,
2. $d(x,y) = d(y,x)$,
3. The triangle inequality must hold $d(x,y) + d(y,z) \geq d(x,z)$.



We will mainly consider the Euclidean metric (L_2-metric).

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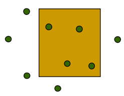
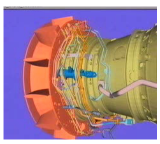
Why computational geometry?

Certain problems are inherently geometric:
- Point location
- Range searching
- Motion planning
- …





Other problems are just much easier to solve if we use the underlying metric:
- Travelling Salesman Problem
- Nearest neighbour
- …

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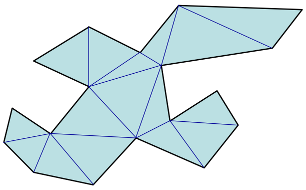
Computational Geometry

Polygon Triangulation and The Art Gallery Problem

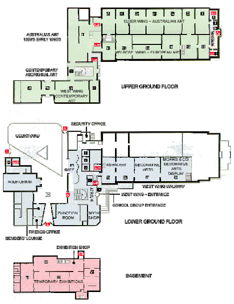


The Art Gallery problem

Question:

How many guards are needed to guard an art gallery?

Victor Klee posed this problem to Václav Chvátal in 1973.

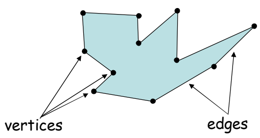


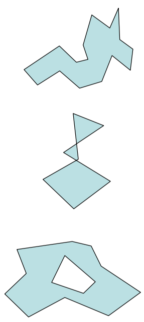
Art Gallery - definitions

Input: An Art Gallery =
A simple polygon with n line segments

Polygon: A region of the plane bounded by a set of straight line segments forming a closed curve.

Simple: no self-intersection, no holes



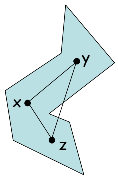


Art Gallery

Guard: (camera, motion sensors, fire detectors, …)

- 2π range visibility
- infinite distance
- cannot see through walls
- cannot move

point a can see point b
iff $(a,b) \subseteq P$



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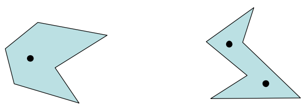
Art Gallery

Question: How many guards are needed to guard an art gallery?

n=6



G(n) = the smallest number of guards that suffice to guard any polygon with n vertices.

Is $G(n) \leq n$? If we place one guard on each vertex?

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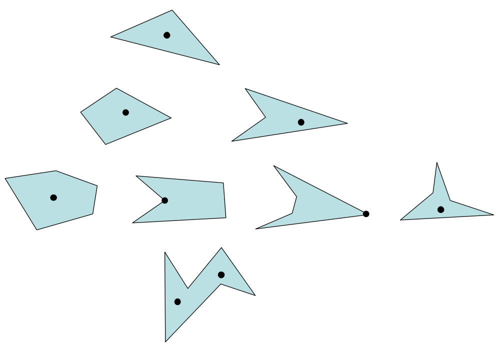
A lower bound

n=3

n=4

n=5

n=6



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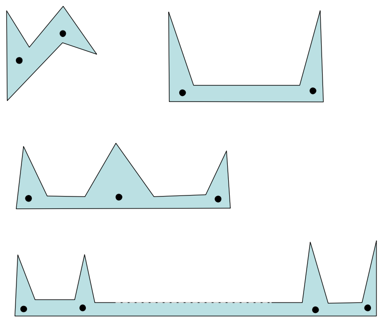
A lower bound

n=6

n=9

n=3k



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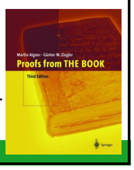
A conjecture

We have shown: $G(n) \geq \lfloor n/3 \rfloor$

Conjecture: $G(n) = \lfloor n/3 \rfloor$
Proved by Chvátal in 1975.

"Steve Fisk learned of Klee's question from Chvátal's article, but found the proof unappealing. He continued thinking about the problem and came up with a solution while dozing off on a bus travelling somewhere in Afghanistan."

An elegant proof was given by Fisk in 1978.
Included in "Proofs from the book".



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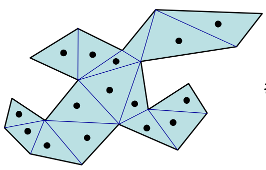
A first upper bound

Prove that $G(n) \leq n-2$.

- Decompose P into pieces that are easy to guard. For example triangles!
- How can we use a triangulation of P to place a set of guards?



#guards = #triangles

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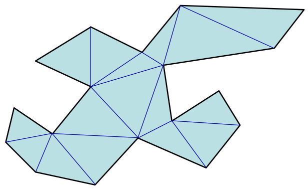
Triangulation

A triangulation can be obtained by adding a maximal number of non-intersecting diagonals within P.

A diagonal of P is a line segment between two vertices of P that are clearly visible to each other.



BUT!

1. Does a triangulation always exist?
2. What is the number of triangles?

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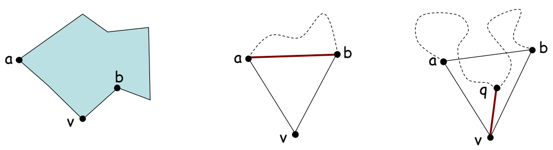
Number of triangles

Does a triangulation always exist?

Is there always a diagonal?

Lemma: Every simple polygon with >3 vertices has a diagonal.



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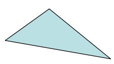
Number of triangles

Theorem: Every simple polygon admits a triangulation.

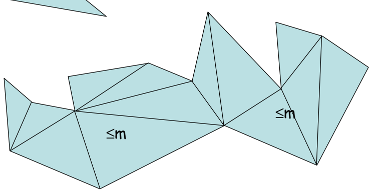
Proof by induction (over the number of vertices):

Base case:
n=3



Induction hyp.:
n≤m

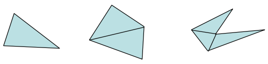
Induction step:
n=m+1



Number of triangles

Theorem: Every triangulation of P of n vertices consists of x triangles.

What's x?

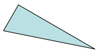


Conjecture: x = n-2

Number of triangles

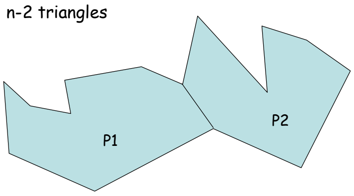
Proof by induction:

Base case (n = 3):



Ind. hyp. (n ≤ m): n-2 triangles

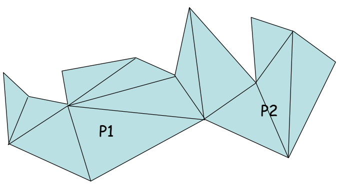
Ind. step (n = m+1):



Number of triangles

|P1| = m1
|P2| = m2

m1+m2 = n + 2

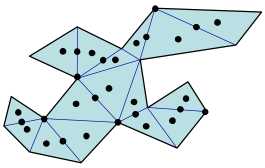


According to ind. hyp.
#triangles in P1 is m1-2
#triangles in P2 is m2-2

(m1-2) + (m2-2) = m1 + m2 - 4 = n – 2 QED

Theorem: Every triangulation of P consists of n-2 triangles.

Back to the Art Gallery



$G(n) \geq \lfloor n/3 \rfloor$

$G(n) \leq$ #guards = #triangles = n-2

How do we place the guards?

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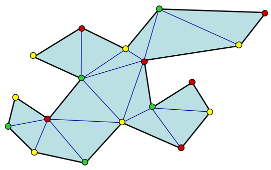
3-colouring

Idea: Assign a colour to each vertex such that no two adjacent vertices have the same colour.

#yellow = 7
#green = 5
#red = 6



Place guards on the green vertices.

$\Rightarrow$ #guards $\leq \lfloor n/3 \rfloor$

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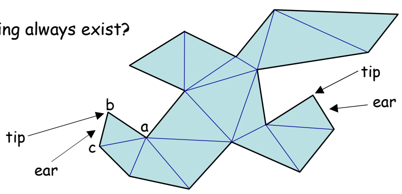
3-colouring

Does a 3-colouring always exist?



Definition: Three consecutive vertices a, b and c of a triangulation form an ear of P if ac is a diagonal of P, where b is the ear tip.

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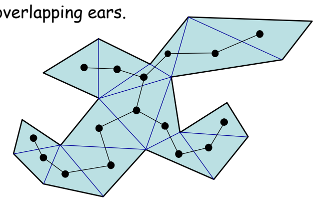
3-colouring

Theorem: Every polygon with n>3 vertices has at least two non-overlapping ears.

Consider the dual D(T) of the triangulation T.



D(T) is a binary tree.

Every tree with at least two nodes has at least two nodes of degree 1 $\Rightarrow$ T has at least two ears.

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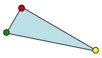
3-colouring

Theorem: The triangulation of a simple polygon can always be 3-coloured.

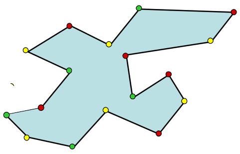
Proof by induction:

Base case (n=3):



Ind. hyp. (n≤m):

Ind. step (n=m+1): Polygon has an ear.



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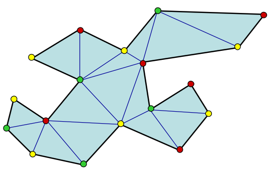
Main results

Theorem:

1. Every simple polygon can be triangulated.
2. Every simple polygon with n vertices can be guarded with $\lfloor n/3 \rfloor$ guards.



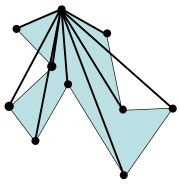
A triangulation exists but how can we compute it?

Triangulation algorithm 1

Theorem: Every polygon has a diagonal.

Testing a diagonal? O(n)

Algorithm 1:
    while P not triangulated do
        (x,y) := find_valid_diagonal(P)
        output (x,y)



Time complexity:
    #iterations = O(n)
    #diagonals = $O(n^2)$
    Test a diagonal = O(n)   ⇒ $O(n^4)$

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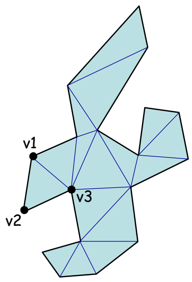
Triangulation algorithm 2

Theorem: Every polygon has at least two non-overlapping ears.

Algorithm 2:

while n > 3 do	n-3
locate a valid ear tip v2	$O(n^2)$
output diagonal (v1,v3)	O(1)
delete v2 from P	O(1)
	Total: $O(n^3)$



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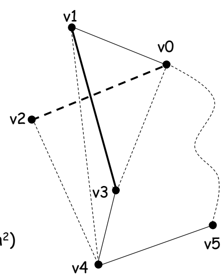
Triangulation algorithm 3

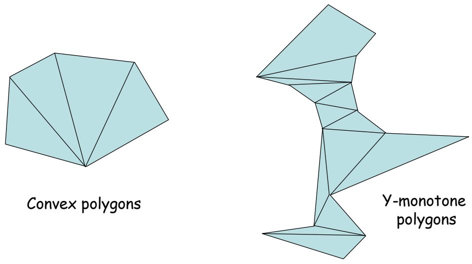
Algorithm 3:

compute all valid ears S	$O(n^2)$
while n > 3 do	n-3
locate a valid ear tip v2	O(1)
output diagonal (v1,v3)	O(1)
delete v2 from P	O(1)
delete (v0,v1,v2) from S	O(n)
delete (v2,v3,v4) from S	O(n)
check ear (v0,v1,v3)	O(n)
check ear (v1,v3,v4)	O(n)
	Total: $O(n^2)$



Triangulation algorithm 4

Observation: Some polygons are very easy to triangulate.



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Polygon triangulation 4

Algorithm 4:

- Partition P into y-monotone pieces. O(n log n)
- Triangulate every y-monotone polygon O(n)

Theorem: Every simple polygon can be triangulated in O(n log n) time.

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Polygon triangulation

- O(n log n) time [Garey, Johnson, Preparata & Tarjan'78]
- O(n loglog n) [Tarjan & van Wijk'88]
- O(n log* n) [Clarkson et al.'89]
- O(n) [Chazelle, Tarjan & van Wijk'91]
- O(n) randomised [Amato, Goodrich & Ramos'00]

Open problem: Is there a simple O(n)-time algorithm?