# COMP5045 Computational Geometry

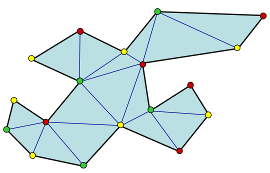


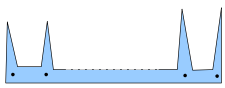




## Last week

- Every simple polygon with n vertices can be decomposed into n-2 triangles.
- Every triangulated simple polygon can be 3-colourable.
- Every simple polygon can be "guarded" by n/3 guards, and n/3 guards is sometimes necessary.
- To find a guard set our algorithm requires a triangulation.
  Last week: $O(n^2)$ time



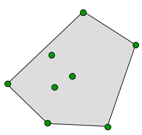






## Convex hull

# Convex hulls and the sweep line technique



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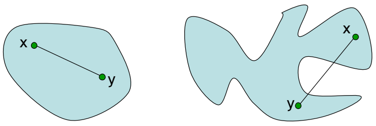
## Definition

A subset S of the plane is convex if for every pair of points x,y in S the straight line segment xy is completely contained in S.



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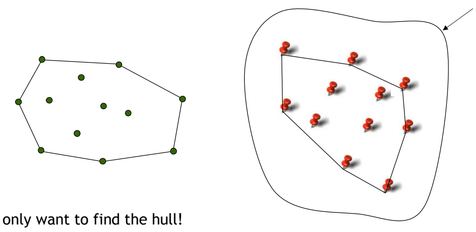
## Definition

The convex hull of a point set S is the smallest convex set containing S.




We only want to find the hull!



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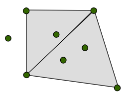
## CH algorithm - 1

Definition:
The CH of a set S of points in d dimensions is the union of all convex combinations of (d+1) points of S.

d=2: Convex combination of 3 points ⇒ a triangle!

Definition implies an algorithm:
A point that does not lie within any triangle of S is a CH vertex.



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## CH algorithm - 1

Algorithm CH1(S)

```
1. for every possible triple of points x,y,z in S do
2.     for every point p in S do
3.        if p lies within the triangle (x,y,z) then
4.              discard p from S
```

Time complexity?

Step 1 is performed $O(n^3)$ times
Step 2 is performed n times/iteration
Step 3 and step 4 cost O(1) /iteration

Total time: $O(n^4)$

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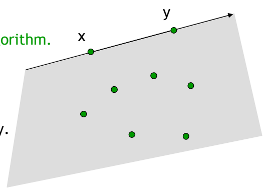
## CH algorithm - 2

Definition:
The CH of S is the intersection of all halfspaces that contain S.

This definition implies a second algorithm.

Consider an edge xy of CH(S).

All points of S must lie to the right of the directed line through x and y.

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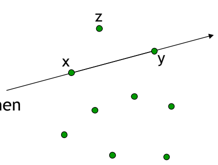
## CH algorithm - 2

Algorithm CH2(S)

```
1. for every ordered pair x,y in S do
2.     valid ← true
3.     for every point z in S-{x,y} do
4.           if z lies to the left of xy then
5.              valid ← false
6.     if valid then
7.        add xy to CH
8. Sort the edges in CH
```




Time complexity?

Steps 1-2, 6-7 : $O(n^2)$ times
Steps 3-5 : (n-2) times/iteration
Step 8 : O(n log n)

Total time: $O(n^3)$

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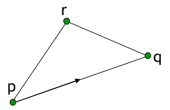
## Check left turn a primitive?

How can we check if a point r lies to the left of a line pq?

$\Rightarrow$ Triangle $\Delta(p,q,r)$ is oriented counter-clockwise.




$p=(p_x,p_y)$, $q=(q_x,q_y)$ and $r=(r_x,r_y)$

$$CCW(p,q,r) = \begin{vmatrix} p_x & q_x & r_x \\ p_y & q_y & r_y \\ 1 & 1 & 1 \end{vmatrix}$$

$$= (q_x - p_x)(r_y - p_y) - (r_x - p_x)(q_y - p_y)$$ [ 2 multiplications, 5 subtractions ]

$\Delta(p,q,r)$ is oriented counter-clockwise iff $CCW(p,q,r) > 0$.

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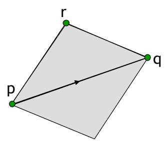
## Check left turn a primitive?

What is CCW(p,q,r)?

$|CCW(p,q,r)| =$




If pqr is a left turn then $CCW(p,q,r) > 0$

If pqr is a right turn then $CCW(p,q,r) < 0$

What about rounding errors?

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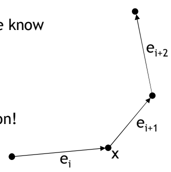
## CH algorithm 3

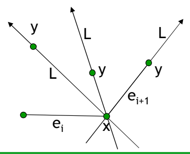
Can we compute the CH faster? Is there anything we know about the CH that we haven't used?




The edges in the CH are linked into a convex polygon!

If we found an edge on the CH with endpoint at x then the next edge must start at x.

Idea:
Draw a line L through x and a point y. Are there any points to the right of L?
If not (x,y) is an edge of CH.




Start point?

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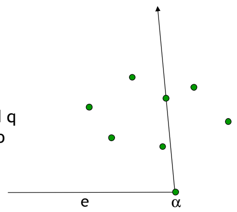
## CH algorithm 3

Algorithm CH3(S)

1. find lowest point p in S
2. α ← p
3. repeat
4. for every point q in S-{α} do
5. valid ← true
6. L ← directed line through α and q
7. for every point r in S-{α,q} do
8. if r to the right of L then
9. valid ← false
10. if valid then
11. add αq to CH
12. α ← q
13. until α == p




Time complexity: $O(n^3)$



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## CH algorithm 3

Algorithm CH3(S)

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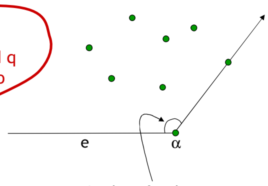
Can this be done faster?




Looking for the point with largest angle!



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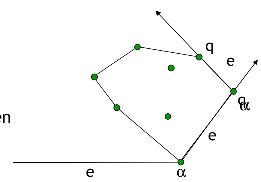
## CH algorithm 4

Algorithm CH4(S)

1. find lowest point p in S
2. α ← p
3. e ← ( $(-\infty, p_y)$ , p )
4. repeat
5. maxAngle ← 0
6. for every point q in S-{α} do
7. if ∠( e , (α,q) ) > maxAngle then
8. nextPoint ← q
9. maxAngle ← ∠( e , (α,q) )
10. e ← ( α , nextPoint)
11. α ← nextPoint
12. until α == p

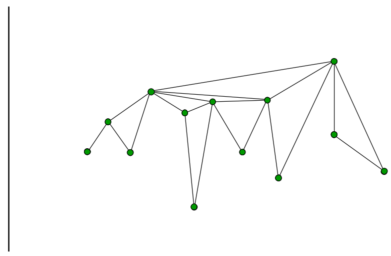



Time complexity: $O(n^2)$

What if the number of points on the CH is small?

## CH algorithm 5 - Incremental

Idea: Maintain hull while adding the points one by one, from left to right
⇔ sweep a line over the points from left to right





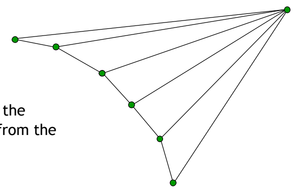


## CH algorithm 5 - Incremental

Running time?

O(n) per insertion ⇒ $O(n^2)$ in total

Can it be that bad?



A point is only added to the hull once and removed from the hull at most once!





## CH algorithm 5 - Incremental

Algorithm CH5(S)

1. sort the points in S from left to right $\langle p_1, p_2, \dots, p_n \rangle$
2. $L_{upper} \leftarrow \langle p_1, p_2 \rangle$
3. for $i \leftarrow 3$ to n do
4. append $p_i$ to $L_{upper}$
5. while $|L_{upper}| > 2$ and the last three points (a,b,c) turn left do
6. Delete b from $L_{upper}$
7. $L_{lower} \leftarrow \langle p_1, p_2 \rangle$

…

13. $L \leftarrow$ join($L_{upper}$, $L_{lower}$)
14. return L

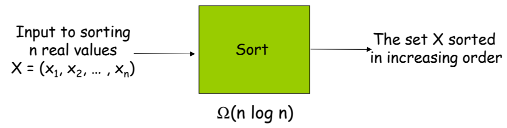
Time complexity: O(n log n)

## Lower bound

Can we do better than O(n log n)?

Prove a lower bound! Use a reduction from Sorting.

Sorting = Ω(n log n) in the algebraic decision tree model







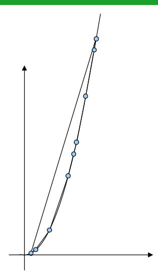

## Lower bound

For each value $x_i$ in X construct a point
$p_i = (x_i, x_i^2)$

P = ($p_1$, $p_2$, … , $p_n$)

Compute CH of P

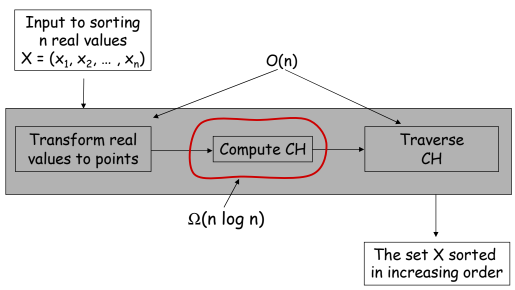
Find the leftmost point p in the CH.
Traverse the CH counter-clockwise from p and output the vertices in the order they are encountered. -> Points in sorted order!







## Lower bound



## What is known?

Preparata & Hong''77 O(n log n)

Kirkpatrick & Seidel'86 O(n log h)

Dynamic convex hull
Brodal & Jacob'02 O(log n) time/update

[This was an open problem since 1981.]

d dimensions
Chazelle'93 $\Theta(n^{\lfloor d/2 \rfloor})$ (complexity of CH)