Due: Friday the 28th of May 2010 at 10am sharp!

Please submit as PDF by email to joachim.gudmundsson@nicta.com.au

As a first step go to `http://gudmundsson.biz/comp5045/course_information.htm` and read the sections: **Advice on how to do the home work** and **Academic Dishonesty**.

COMP 5045 – Assignment 3

1. You are given a collection of vertical line segments in the top-right quadrant of the x, y plane. Each line segment has one endpoint on the x-axis and the other endpoint has a positive y-coordinate. Imagine that from the top of each segment a horizontal bullet is shot to the left. The problem is to determine the index of the segment that is first hit by each of these bullet paths. If no segment is hit, return the index 0, as illustrated in Fig. 1a. The input is a sequence of top endpoints of each segment $p_i = (x_i, y_i)$, for $1 \le i \le n$, which are sorted in increasing order by x-coordinate. The output is the sequence of indices, one for each segment. Present an $O(n)$ time algorithm to solve this problem. Explain how your algorithm works and justify its running time. [10 points]

2. Let P be a convex polygon with n vertices, where the vertices are given in counter-clockwise order. Consider a direction d. If you are standing below P (where below refers to the direction d at infinity) and look in direction d, then you will see a certain number of edges of P. In the example shown in Fig. 1b the point can see 3 edges, however, there exists a direction where a point can see 4 edges. Give an $O(n)$ time algorithm that computes a direction d for which the number of edges that are visible is maximized. Prove complexity and correctness. [10 points]

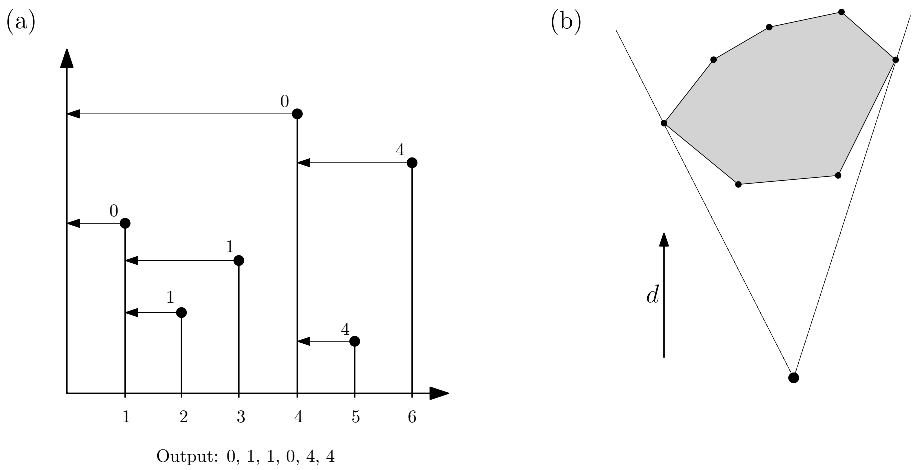

Figure 1: Illustrations of questions 1 and 2.

3. Let P be a set of n points in the plane, give an $O(n^2)$ time algorithms to find a line containing the maximum number of points in P. Argue for its correctness and complexity. [10 points]

4. Let P be a set of n points in the plane. Construct a data structure that can answer empty circle queries efficiently. An empty circle query for a query point q asks for the largest circle with centre at q that does not contain any points of P. For full points the preprocessing should be done in $O(n \log n)$ expected time using $O(n)$ expected space and the queries should be answered in $O(\log n)$ expected time. Argue its correctness and complexity. [10 points]

5. Let H be a set of n disjoint horizontal line segments in the plane, and let V be a set of m disjoint vertical line segments in the plane. Describe an $O((n+m)\log(n+m))$ time algorithm to count the number of intersections among $H \bigcup V$. Argue its correctness and complexity. [10 points]

6. The goal of this problem is to construct a t-spanner. Let P be a set of n points in the plane and let $t > 1$ be the desired stretch factor. Consider the set S of all possible segments between points in P.

   Consider the following construction of a graph G of P (see also pseudocode below). Initially set $G = (P, \emptyset)$, that is, vertex set is the points in P and it has no edges. Process the segments in $S = \langle s_1, s_2, \ldots, s_{\binom{n}{2}} \rangle$ in order of increasing length as follows. When a segment $s_i = (u, v)$ is processed a shortest path query is performed between u and v in the current graph G. If this shortest path is at most $t \cdot |uv|$ then s_i is discarded and the next segment is processed. Otherwise, if the shortest path in G is greater than $t \cdot |uv|$ then the edge (u, v) is added to G before the next segment is processed. When all the segments in S have been processed we are done and the resulting graph is denoted $G = (P, E)$.

   Algorithm SPANNER(P)

   ```
   S := set of all (n choose 2) possible segments between points of P
   sort the segments in S by increasing weight ⟨s_1, s_2, ..., s_(n choose 2)⟩
   E := ∅
   G := (P, E)
   for each segment s_i = (u, v) ∈ S do
           if ShortestPath(G, u, v) > t · |u, v| then
                   E := E ∪ {(u, v)}
                   G := (P, E)
   output G
   ```

   (a) Prove that G is a t-spanner. (Hint: Use proof of contradiction.) [3 points]

   (b) The girth of a graph is the number of edges of a shortest cycle contained in the graph. It is well-known that any graph with girth at most g has $O(n^{1+2/(g-2)})$ edges. Prove that G has girth at least $t+1$, which then also implies that G has $O(n^{1+2/(t-1)})$ edges. (Hint: Prove that there are no cycles in the graph of length less than $t+1$.) [6 points]

   (c) Can this approach be generalized to any graph, that is, not necessarily geometric graphs? Argue why or why not. [1 point]